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Slope Stability Analysis Based upon the Linear Matching Method

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Abstract: **This paper describes a technique for computing optimal upper bound slopes stability factors based upon the Linear Matching Method. The method assumes a perfectly plastic material which obeys the Drucker-Prager yield criterion and its associated flow rule and involves the sequential matching of an appropriate linear material to the yield function. The basic concepts are described and the convergence of this method is guaranteed whether the real material is matched to a compressible or incompressible linear material. The process converges to the least upper bound associated with the class of displacement fields when implemented in a finite element method. Comparison with published solutions illustrates the accuracy and feasibility of the proposed method for a simple homogeneous slope stability problem.**

Keywords: **Slope stability, Limit analysis, Linear Matching Method, Finite Element Analysis, Associated flow rule.**

I. INTRODUCTION

Predicting the stability of soil slopes is a classical problem for geotechnical engineers when designing embankments for roads, railways and other engineering structures. A good understanding of failure mechanism of both natural and man-made earthen slopes is therefore necessary in order to better assist any remediation measure of potentially moving or failing slopes. Slope stability analysis methods have been developed over many years and continue to develop as a result of considerable advancement in theories, new constitutive soil models, instrumentations and technology. Many researchers have focused on assessing the stability of slopes using different methods of analysis. The limit equilibrium method and limit analysis based upon the upper and lower bound theorems which enable the true collapse load to be bracketed from above and below have been widely used as engineering design tools to predict the safety of slopes. Although the limit equilibrium analysis may sometimes lead to significant errors as both kinematic and static admissibility are violated, this approach enables complex soil profiles, seepage and a variety of loading conditions to be easily dealt with. Many comparisons of limit equilibrium (e.g. Fredlund and Krahn [1], Duncan and Wright [2], Nash [3]) indicate that techniques that satisfy all conditions of global equilibrium give similar results. Of particular interest in this article is the kinematic approach of limit analysis with application to 2D failures. An early application of this method was shown by Drucker and Prager [4], Chen *et al.* [5] and Chen [6], who considered a slope failing under plane-strain conditions. In recent years the emphasis has increasingly turned towards computational methods that rely upon plasticity limit theorems to locate the most critical failure surface by using various search strategies. These provide, assuming an associate flow rule, strict upper and lower bound values using the kinematics of the displacement finite element method for upper bounds Chen *et al.* [7] and stress finite elements for lower bounds Yu *et al.* [8], Kim *et al.* [9], Li *et al.* [10]. The relevant limit theorem is posed as a programming problem, to which an appropriate programming method is applied to estimate rigorous lower and upper bound solutions for the stability of simple slopes in both homogeneous and inhomogeneous soils. Among other available analysis methods are the boundary element methods Jiang [11] and the finite element methods, Matsui & San [12].

In parallel with these developments, limit analysis and its extension "the shakedown analysis" have been applied in the prediction of the deformation and life assessment of structures subjected to cyclic mechanical and thermal loadings. In previous papers, (Boulbibane and Ponter [13], Boulbibane and Ponter [14]), a procedure was described for the evaluation of limit loads related to certain design criteria for structural materials in a cyclic state of creep and limit loads for indentation problems using the Drucker-Prager yield criterion, respectively. The procedure was based upon the Linear Matching Method where a sequence of linear problems are solved with spatially varying linear moduli. By repeated iterations the upper bound reduces to the minimum upper bound associated with the class of displacement field of the finite element mesh. For yield conditions dependent on both the effective von Mises stress and the hydrostatic pressure, both shear and bulk moduli of the linear material are adjusted so that, for the current strain rate distribution, the linear material is matched to the yield condition. A sufficient condition for convergence is then given by the condition that the surface of constant complementary energy lies entirely outside the yield surface. This condition cannot, however, be always satisfied. In Boulbibane and Ponter [15], it was shown that convergence occurred even if the sufficient condition was not satisfied. By considering a class of elliptic yield conditions (in von Mises effective stress-pressure space), it was possible to compute solutions where the sufficient condition is both satisfied and not satisfied by varying the effective Poison's ratios of the linear material.

In this paper we discuss the application of the Linear Matching Method to define upper bound limits for slope stability factors. In particular, we investigate the influence of key mechanical and geometrical parameters on the least upper bound solutions. In order to invoke the upper bound theorem of classical plasticity theory, a perfectly plastic soil model is assumed, which may be either purely cohesive or cohesive-frictional. The Drucker-Prager yield condition is adopted assuming an associated flow rule. The method calls upon procedures which form the basis of linear finite element analysis, it is possible to implement the method through the optional user procedures which are often included in commercial codes. For the solutions described in this paper the general code ABAQUS is used. In addition, a comparison with other methods is given and final remarks complete the paper.

II. METHOD DESCRIPTION

In the present study, it is assumed that the yield function $f(\sigma)$ is defined by the Drucker-Prager failure criterion. Following the sign convention usually adopted in soil mechanics, according to which compression is considered positive, this function can be written in terms of the hydrostatic pressure p and the von Mises effective stress $\bar{\sigma}$ as follows:

$$f(\sigma) = \bar{\sigma} - \left[\bar{c} - p \tan \bar{\phi}\right] = 0 \tag{1}$$

with $$p = \frac{\sigma_{kk}}{3} = \frac{I_1}{3} \quad , \qquad \bar{\sigma} = \sqrt{3/2\,\sigma'_{ij}\sigma'_{ij}} = \sqrt{3}J_2^{1/2} \tag{2}$$

are related to the first stress tensor invariant I_1 and to the second deviator stress tensor invariant J_2, which depends on the stress deviator σ'_{ij} defined by:

$$\sigma'_{ij} = \sigma_{ij} - p\delta_{ij} \tag{3}$$

The cohesion $\bar{c}$ and the angle of internal friction $\bar{\phi}$ are assumed as the two material parameters defining the yield surface. The plastic strain rates resulting from yield may be represented as a vector $(\dot{\bar{\varepsilon}}^p, \dot{\varepsilon}_v^p)$, where $\dot{\bar{\varepsilon}} = \sqrt{\tfrac{2}{3}\dot{\varepsilon}'_{ij}\,\dot{\varepsilon}'_{ij}}$ denote the von Mises effective strain rate and $\dot{\varepsilon}_v = \dot{\varepsilon}_{kk}$ denotes the volumetric strain rate.

For an arbitrary yield condition the associated flow rule is given by;

$$\dot{\varepsilon}_{ij}^p = \dot{\Gamma}\frac{\partial f}{\partial \sigma_{ij}} \tag{4}$$

where $\dot{\Gamma}$ is a plastic multiplier. When applied to the Drucker-Prager yield condition as given by equation (1), equation (4) may be written in the form:

$$\dot{\bar{\varepsilon}}^{p} = \dot{\Gamma}\frac{\partial f}{\partial \bar{\sigma}} = \dot{\Gamma} \quad \text{and} \quad \dot{\varepsilon}_{v}^{p} = \dot{\Gamma}\frac{\partial f}{\partial p} = \dot{\Gamma}\tan\bar{\phi} \tag{5}$$

and hence,

$$\dot{\varepsilon}_{v}^{p} = \dot{\bar{\varepsilon}}^{p}\tan\bar{\phi} \tag{6}$$

Values of the strength parameters $\bar{c}$ and $\bar{\phi}$ for particular soils are often not directly available. Instead, the user is provided with the friction angle ϕ and cohesion c values of the Mohr-Coulomb model:

$$f(\sigma) = (\sigma_1 - \sigma_2) + (\sigma_1 + \sigma_2)\sin\phi - 2c\cos\phi = 0 \tag{7}$$

where $\sigma_1 \geq \sigma_3 \geq \sigma_2$ are the maximum, intermediate and minimum principal stresses. The two models equations (1, 7) become identical in terms of in-plane quantities for conditions of plane strain and this leads to the following equations:

$$\tan\bar{\phi} = \frac{\sqrt{3}\sin\phi}{\sqrt{1+\frac{1}{3}\sin^2\phi}} \quad , \quad \frac{\bar{c}}{c} = \frac{\sqrt{3}\cos\phi}{\sqrt{1+\frac{1}{3}\sin^2\phi}} \tag{8}$$

A. The Linear Matching Method

As mentioned previously, the method involves a programming technique where, at each iteration, the moduli of a linear viscous material are adjusted so that, for the current strain rate distribution, the linear material is matched to the yield condition. However, this condition alone is insufficient to uniquely define the moduli or for convergence to occur and additional conditions need to be applied. Detailed derivations for the general limit analysis problem are discussed in (Boulbibane and Ponter, [15] and Ponter *et al.*, [16]). Here we specialise this formulation to upper bound limits for slope stability factors.

An isotropic linear viscous material is described in terms of a shear modulus μ, bulk modulus K and an initial pressure p_L;

$$\bar{U}(\bar{\sigma}, p) = \frac{1}{6\mu}\bar{\sigma}^2 + \frac{(p+p_L)^2}{2K} \tag{9}$$

$$\dot{\bar{\varepsilon}} = \frac{\partial \bar{U}}{\partial \bar{\sigma}} = \frac{\bar{\sigma}}{3\mu}, \quad \dot{\varepsilon}_v = \frac{\partial \bar{U}}{\partial p} = \frac{(p+p_L)}{K} \tag{10}$$

where $\bar{U}$ defines the complementary dissipation rate of the linear material.

The method consists of the generation of a sequence of linear problem for such a material where μ, K and p_L vary spatially. At a certain stage of an iterative process, described below, a compatible strain rate distribution $\dot{\varepsilon}_{ij}^{i}$ and pressure distribution p^i are derived. The linear coefficients are then chosen so that the linear material and the yield condition are both consistent with $\dot{\varepsilon}_{ij}^{i}$ (Ponter *et al.*, [16]). For the Drucker-Prager yield condition with an associated flow rule, this would require that equation (6) holds. However, it is not generally satisfied by the current linear solution. As a result, in the matching condition $\dot{\bar{\varepsilon}}^{i}$ and p^i are used but $\dot{\varepsilon}_v^{i}$ is not used (see Boulbibane and Ponter [15]). Hence if $\bar{\sigma}_y^i$ and p_y^i satisfy the yield condition (1) at the matching condition, the linear moduli are required to satisfy the following matching equations,

$$\dot{\bar{\varepsilon}}^{i} = \frac{\bar{\sigma}_y^i}{3\mu} \quad \text{where} \quad p_y^i = p^i, \quad \bar{\sigma}_y^i = \bar{c} - p^i\tan\bar{\phi} \tag{11}$$

and

$$\dot{\varepsilon}_v^{*} = \frac{(p_y^i + p_L)}{K} \quad \text{with} \quad \dot{\varepsilon}_v^{*} = \dot{\bar{\varepsilon}}^{i}\tan\bar{\phi} \tag{12}$$

The matching condition equations (11) and (12) provides two equations for the three independent material constants, μ, K and p_L or, equivalently E, ν_L and p_L, where ν_L is a (viscous) Poisons ratio and E a uniaxial modulus. Hence one of these three may be chosen. In the following we assume throughout the iterative process that $\nu_L = const$ and hence E and p_L are given by;

$$E = \frac{2(1+\nu_L)}{3}\frac{\bar{\sigma}_y^i}{\bar{\dot{\varepsilon}}^i}, \qquad p_L = -p_y^i + K\bar{\dot{\varepsilon}}^i \tan\bar{\phi} \tag{13}$$

In a typical slope stability analysis the only load considered is that of the soil weight. This load is given in terms of the unit weight γ, and the limit analysis problem can be stated in the following manner: find the magnitude of unit weight that will cause the slope of given geometry to fail. Considering that the slope also may be loaded with a given distributed load (traction) P_i on boundary S, the principle of virtual work written for the true (but unknown) stress field σ_{ij}^c produced by the soil weight γ_i, on a compatible strain rate field $\dot{\varepsilon}_{ij}^c$ with corresponding displacement rate field $\dot{u}_i^c$, takes the form:

$$\lambda_{UB}^c\left[\int_S P_i\dot{u}_i^c dV + \gamma\int_V n_i\dot{u}_i^c dV\right] = \int_V \sigma_{ij}^c\dot{\varepsilon}_{ij}^c dV \quad , \qquad \lambda_{UB}^c \ge \lambda_L \tag{14}$$

Here λ_{UB}^c denotes the upper bound load parameter, γ the unit weight magnitude and n_i the unit vector in the direction of gravity. Assuming the associated flow condition, equation (6), and applying the principle of Virtual Work, Eq. (14) may be written in the equivalent form,

$$\lambda_{UB}^c\int_V \sigma_{ij}^P\dot{\varepsilon}_{ij}^c dV = \int_V (\bar{\sigma}_y^c + p_y^c\tan\bar{\phi})\bar{\dot{\varepsilon}}^c dV \tag{15}$$

where σ_{ij}^P may be any distribution of stress in equilibrium with P_i and soil weight γ_i. For consistency with the assumptions above, in evaluating λ_{UB}^c from (15) a distribution of $\dot{\varepsilon}_{ij}^c = \dot{\varepsilon}_{ij}^{iE}$ is used that is derived from $\dot{\varepsilon}_v$ given by the flow rule (6);

$$\dot{\varepsilon}_{ij}'^i = \dot{\varepsilon}_{ij}^i - \dot{\varepsilon}_{kk}^i\delta_{ij}/3 \qquad \text{and} \qquad \dot{\varepsilon}_{ij}^{iE} = \dot{\varepsilon}_{ij}'^i + \bar{\dot{\varepsilon}}^i\tan\bar{\phi}\delta_{ij}/3 \tag{16}$$

In this case Eq. (15) is no longer a strict upper bound as $\dot{\varepsilon}_{ij}^{iE}$ is not necessarily a compatible strain field. However it may still be regarded as an energy balance equation and can be used to calculate an estimate of the unit weight causing failure, $\lambda_{UB}^c\gamma$ which will be not less than the true value of $\lambda_L\gamma$ causing the slope to collapse. Because $\gamma h/c$ is a dimensionless group in the problem, the inequality (14) can be multiplied by the true value of $\gamma h/c$ to yield:

$$\frac{\lambda_{UB}^c\gamma h}{c} \ge \frac{\lambda_L\gamma h}{c} \tag{17}$$

With these definitions a linear problem for a new compatible strain rate history $\dot{\varepsilon}_{ij}^f$ and equilibrium stress field σ_{ij}^{pf} is given by:

$$\dot{\varepsilon}_{ij}'^f = \frac{1}{2\mu}\lambda_{UB}^i\sigma_{ij}^{pf\prime} \tag{18}$$

$$\dot{\varepsilon}_{kk}^f = \frac{1}{3K}\left(\lambda_{UB}^i\sigma_{kk}^{pf} + p_L\right) \tag{19}$$

The resulting value of λ_{UB}^f, obtained by substituting the solution of (18), (19) into (15), may be expected to be a closer approximation to the optimal value than λ_{UB}^i. An initial solution assumes that values of the moduli μ, K and p_L are chosen arbitrary. As a result of this initial solution, the iterative process described by equations (11 – 15) is repeatedly applied until convergence occurs, i.e. $\dot{\varepsilon}_{ij}^f = \dot{\varepsilon}_{ij}^i$ and $\lambda_{UB}^f = \lambda_{UB}^i$. The plastic strain components at instants where there is no strain in the converge solution then decline in relative magnitude until they make no contribution to the upper bound solution. Although the sufficient condition for convergence in Ponter *et al.* [16] is not satisfied, least upper bounds are still obtained for the problems discussed in this paper, provided convergence occurs. This point was clarified through numerical examples in Boulbibane and Ponter [14].

The method is implemented in the commercial finite element code ABAQUS. The normal mode of operation of such codes for non-linear analysis involves the solution of a sequence of linearized problems for incremental changes in stress, stain and displacement in time intervals corresponding to a predefined history of loading. At each increment, user routine allows a dynamic prescription of the Jacobian which defines the relationship between increments of stress and strain. The implementation involves carrying through a standard load history calculation for the body, but setting up the calculation sequence and Jacobian values so that each incremental solution provides the data for an iteration in the iterative process. Volume integral options evaluate the upper bound value of λ which is then provided to the user routine for the evaluation of the next iteration. The matching condition (11, 12) is then applied at each Gauss point and results in variations of the three independent material constants, μ, K and p_L within each element. At convergence, when consecutive linear solutions are identical, the conditions for the exact solution are obtained. The compatible strain-rate distribution is hence consistent with a stress distribution of the form $\lambda\sigma_{ij}^{p}$ that satisfies the yield condition where the strain rates are non-zero. For finite element solutions where equilibrium of $\lambda\sigma_{ij}^{p}$ is only satisfied in a Rayleigh-Ritz sense, the converged solution satisfies the conditions for the minimum upper bound amongst the class of displacement fields defined by the finite element mesh.

III. APPLICATION OF THE METHOD TO SLOPES STABILITY

The above numerical procedure is used to investigate a simple embankment under its own weight. According to the upper bound theorem of limit analysis the embankment shown in Figure1 will collapse under its own weight if for any assumed failure mechanism the rate of external work done by the soil exceeds the rate of internal energy dissipation. In his limit analysis of a 2D slope collapse, Chen [6] pointed out that in plane-strain analysis, a rotational failure mechanism yields the least best estimate of the dimensionless critical height $\gamma h/c$. The solutions described below are intended as reference solutions that supplement the widely used solution of Chen *et al.* [5]. In this case, the authors consider the slope stability problem using four-noded isoparametric plane strain elements as shown in Fig. 1.

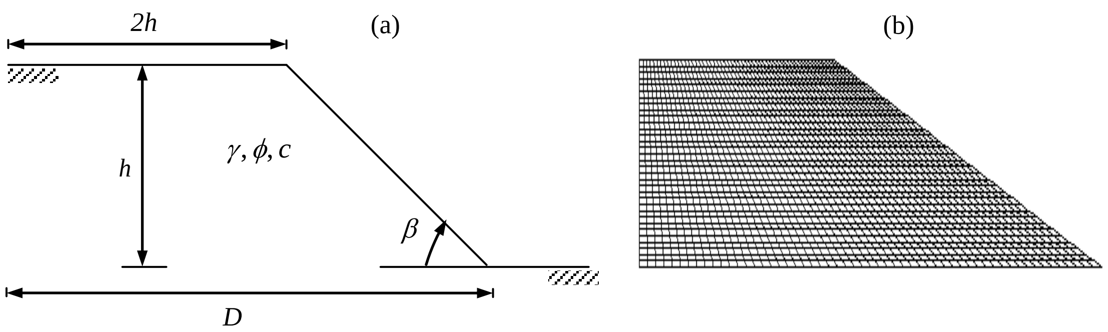

Fig.1 (a) Embankment geometry and (b) finite element mesh

The soil is assumed to be a perfectly plastic material which obeys the Coulomb yield criterion and its associated flow rule. The viscous linear material Poisson's ratio is assumed to be ($\nu_L = 0.35$) for all computations. Values of internal friction given in Table 1 are those associated to Mohr-Coulomb yield condition. However, in the proposed computational technique parameters ($\bar{c}$, $\bar{\phi}$) associated to Drucker-Parger, which can be easily obtained using equation (11), are used. An upper bound solution for the stability factor is sought by optimizing the unit weight of the soil mass with a load multiplier λ for a slope with fixed height, internal friction angle and cohesion. As a result, finding the minimum of the stability factor is equivalent to computing the minimum of the load multiplier λ_{UB}^{c}, as described above. Results in terms of $N_s = \lambda\gamma h/c$ as well as already existing limit analysis results are tabulated numerically in Table 1 for various material properties and slop geometries. As can be noted the proposed technique predicts nearly the same values of N_s as those obtained by Chen *et al.* [5] with the exception for $\phi = 0^\circ$ and $\beta < 30^\circ$. It is found that Values of $N_s = \lambda\gamma h/c$ obtained by Chen for ($\phi = 0^\circ$) do not change as the value of (β) changes. However, we found that these values increase as (β) decreases.

Table 1. Stability factors $N_s = \lambda\gamma h/c$ obtained by the Linear Matching Method and values given in Chen *et al.* [5] by assuming a curved failure mechanism.

Friction angle ϕ°	Slope angle β in degrees							
	20		**30**		**45**		**60**	
	LMM	LA	LMM	LA	LMM	LA	LMM	LA
0	9.51	5.53	7.59	5.53	6.11	5.53	5.25	5.25
5	14.35	11.46	10.09	9.13	7.48	7.35	6.14	6.16
10	26.02	23.14	14.35	13.50	9.45	9.31	7.31	7.26
15	72.0	69.40	22.34	21.69	12.17	12.05	8.70	8.63
20			41.99	41.22	16.43	16.16	10.43	10.39
25					23.28	22.90	13.04	12.74
30					36.75	35.54	16.55	16.04
35							21.60	20.94
40							30.13	28.91

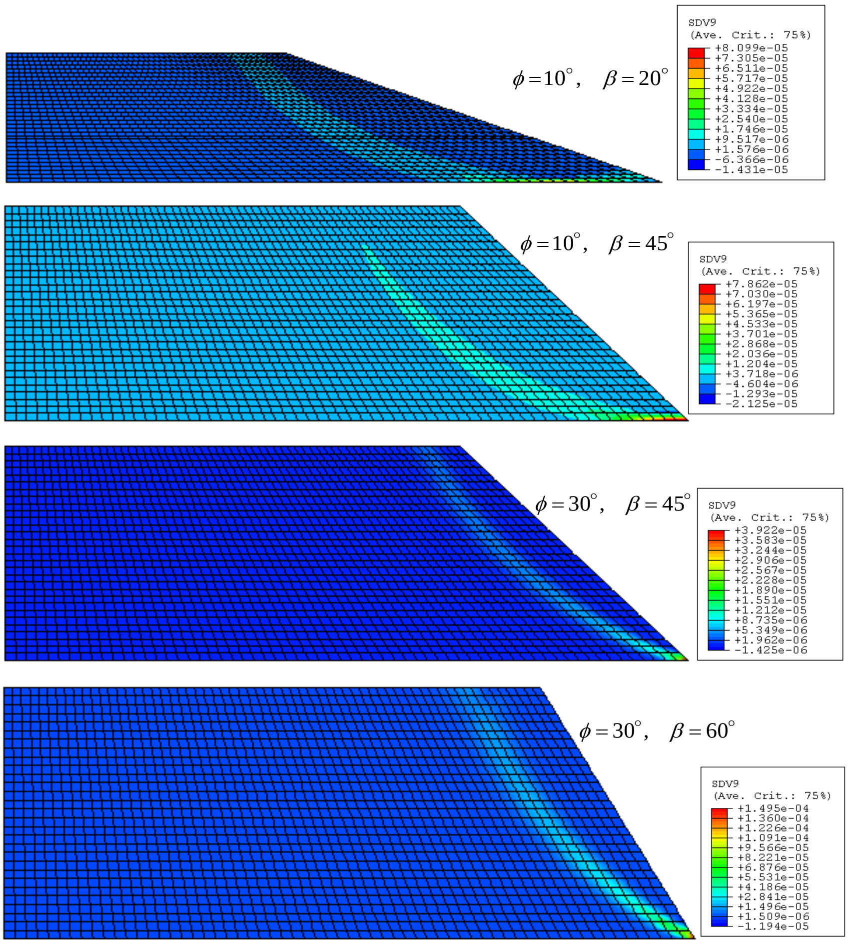

Fig.2 Formation and development of shear zone within the slope - contour plots of von-Mises effective strain.

Using this approach, the response of the soil is effectively captured and the location and propagation of the shear zone is reliably simulated, as shown in Fig. 2. It is interesting to note the evolution of the failure mode with β and ϕ. When β

is large, the failure mode develops locally near the facing and when β is relatively small, slope instability develops through the foundation soil as local surficial or deep seated failure. Figure 3 shows that for a simple homogeneous slope, the stability factors obtained by the proposed method are almost equal to previously published solutions.

IV. CONCLUSION

In this study, a new technique has been proposed to analyse the stability of slopes by convergent non-linear programming method where the local gradient of the upper bound functional and the potential energy of the linear problem are matched at a current strain rate or during a strain rate history. It has been indicated earlier for specific examples that the kinematic approach of limit analysis yields an upper bound to the critical height of the slope. This statement is now proved to be more general, and it follows that the stability factor determined by the LMM approach is also an upper bound on its "true" value. The convergent programming method is capable of producing good estimate to the upper bound critical heights for bodies with isotropic yield condition where the yield function is dependent on the Von Mises effective stress and the hydrostatic pressure and in predicting failure modes. As the method relies upon linear solutions at each iteration, an implementation can be achieved in any commercial finite element package and this could provide an efficient and safe way for practical slope design.

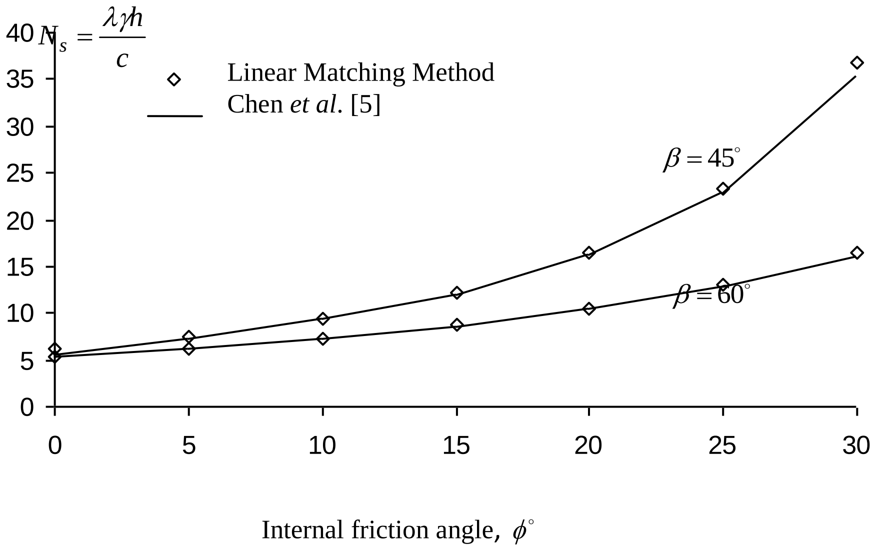

Fig.3 Stability analysis of simple embankment under its own weight: effect of internal frictional angles and slope angles on upper bound of slope critical heights.

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